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ON THE DUAL STIEFEL–WHITNEY CLASSES OF SOME GRASSMANN MANIFOLDS*

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Abstract. We present some non-vanishing dual Stiefel–Whitney classes of the Grassmann manifolds $O(n)/O(4) \times O(n-4)$ for $n = 2^s + 2$ and $n = 2^s + 3$ ($s \geqq 3$), providing a supplement to results of Hiller, Stong, and Oproiu. Some applications are also mentioned.

1. Introduction

When studying some properties of a given smooth connected closed d-dimensional manifold M, it may be useful to know its (total) dual Stiefel–Whitney class, denoted by $\bar{w}(M)$. Actually, this is the Stiefel–Whitney class of the normal bundle of any immersion of M into Euclidean space. If the qth class $\bar{w}_q(M)$ does not vanish, then M does not immerse in $\mathbb{R}^{d+q-1}$ and does not embed in $\mathbb{R}^{d+q}$ (see, e.g., [12, Ch. 16]). Another interesting consequence of $\bar{w}_q(M) \neq 0$ (see [3, Theorem 1.3]) is the nonexistence of $2t$-regular maps from M to $\mathbb{R}^{t(d+1+q)}$ (recall that a continuous map $f : X \to \mathbb{R}^N$ is said to be

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t-regular, if $f(x_1),\dots,f(x_t)$ are linearly independent over $\mathbb{R}$ for any t-tuple $x_1,\dots,x_t$ of distinct points of X).

Let $G_{n,k}$ denote the Grassmann manifold of unoriented k-dimensional vector subspaces in $\mathbb{R}^n$; as a homogeneous space, $G_{n,k}=O(n)/O(k)\times O(n-k)$. In the papers [9], [10], [4], one can find various results concerning non-vanishing dual Stiefel–Whitney classes for several families of $G_{n,k}$. In this paper, we derive new results for $k=4$.

It is known (Borel [2]), that the $\mathbb{Z}_2$-cohomology algebra of the Grassmann manifold is given by

$$H^*(G_{n,k};\mathbb{Z}_2)=\mathbb{Z}_2[w_1,\dots,w_k]/I_{n,k},$$

where $w=1+w_1+w_2+\dots+w_k$ is the total Stiefel–Whitney class of the canonical k-plane bundle γ over $G_{n,k}$, $\bar{w}=1+\bar{w}_1+\bar{w}_2+\dots+\bar{w}_{n-k}$ is the total Stiefel–Whitney class of the complementary bundle $\gamma^\perp$, and $I_{n,k}$ is the ideal generated by the homogeneous components of $(1+w_1+\dots+w_k)^{-1}$ in dimensions $n-k+1,\dots,n$. For the tangent bundle, it is also well known (e.g. [5]) that $TG_{n,k}\oplus\gamma\otimes\gamma\cong n\gamma$. Applying $\gamma\oplus\gamma^\perp\cong\varepsilon^n$, where ε^n is the trivial n-plane bundle, we obtain for the dual Stiefel–Whitney class of $G_{n,k}$ that

$$\bar{w}(G_{n,k})=w(\gamma\otimes\gamma\oplus n\gamma^\perp)=w(\gamma\otimes\gamma)\cdot(1+\bar{w}_1+\dots+\bar{w}_{n-k})^n. \tag{1.1}$$

In general, calculations in the cohomology algebra $H^*(G_{n,k};\mathbb{Z}_2)$ are very complicated. Sometimes Gröbner bases make them easier, as in [8]. In this paper, we shall use Stong's method presented in [11] and, starting from (1.1), we shall show that some high-dimensional dual Stiefel–Whitney classes for the Grassmann manifolds $G_{2^s+2,4}$ and $G_{2^s+3,4}$ ($s\geqq 3$) do not vanish. For later references, we now briefly recall some facts from [11].

Fact (a). The cohomology algebra $H^*\big(\mathrm{Flag}\,(\mathbb{R}^n);\mathbb{Z}_2\big)$, where

$$\mathrm{Flag}\,(\mathbb{R}^n)=O(n)/O(1)\times\ldots\times O(1),$$

can be identified with

$$\mathbb{Z}_2[e_1,\dots,e_n]\Big/\bigg(\prod_{i=1}^n(1+e_i)=1\bigg),$$

where e_i is the first Stiefel–Whitney class of the ith canonical line bundle over $\mathrm{Flag}\,(\mathbb{R}^n)$.

FACT (b). The obvious bundle projection $\pi : \mathrm{Flag}\,(\mathbb{R}^n) \to G_{n,k}$ induces a cohomology monomorphism, $\pi^* : H^*(G_{n,k};\mathbb{Z}_2) \to H^*\big(\mathrm{Flag}\,(\mathbb{R}^n);\mathbb{Z}_2\big)$, and we have

$$\pi^*\big(w(\gamma)\big) = \prod_{i=1}^{k}(1+e_i), \qquad \pi^*\big(w(\gamma^{\perp})\big) = \prod_{i=k+1}^{n}(1+e_i).$$

FACT (c). The value of the class $u \in H^*(G_{n,k})$ on the fundamental class of $G_{n,k}$ is the same as the value of $\pi^*(u)\cdot e_1^{k-1}e_2^{k-2}\dots e_{k-1}e_{k+1}^{n-k-1}e_{k+2}^{n-k-2}\dots e_{n-1}$ on the fundamental class of $\mathrm{Flag}\,(\mathbb{R}^n)$.

FACT (d). The nonzero monomials in $H^{\mathrm{top}}\big(\mathrm{Flag}\,(\mathbb{R}^n)\big)$ are precisely those of the form

$$e_{\sigma(1)}^{n-1}\dots e_{\sigma(i)}^{n-i}\dots e_{\sigma(n)}^{0},$$

i.e. those with no repeated exponents.

2. Results

We first recall results of [10] and [4].

PROPOSITION 2.1 (Oproiu [10, Theorem 1(i)]; Hiller, Stong [4, Proposition 4.1]). *If $k \leqq 2^{s-1} \leqq n-k$ and $n \leqq 2^s$, then $\bar{w}_q(G_{n,k}) \neq 0$ for $q = k(2^s+k-n-1)$.*

PROPOSITION 2.2 (Hiller, Stong [4, Proposition 4.3]). *If $k \leqq 2^{s-1}-1$ and $n = 2^s+1$, then $\bar{w}_q(G_{n,k}) \neq 0$ for $q = 2^s+k^2-2k$.*

Applying this to the case $k=4$, we have the following.

COROLLARY 2.3. *Let $s \geqq 3$.*

(i) *If $2^{s-1}+4 \leqq n \leqq 2^s$, then $\bar{w}_q(G_{n,4}) \neq 0$ for $q = 4\cdot 2^s - 4n + 12$;*

(ii) *if $n = 2^s$, then $\bar{w}_{12}(G_{n,4}) \neq 0$;*

(iii) *if $n = 2^s+1$, then $\bar{w}_q(G_{n,4}) \neq 0$ for $q = 2^s+8$;*

(iv) *if $n = 2^s+4$, then $\bar{w}_q(G_{n,4}) \neq 0$ for $q = 4\cdot 2^s - 4$.*

Indeed, (i) is directly implied by Proposition 2.1. Part (ii) is a special case of (i). For $s > 3$, (iii) is implied by Proposition 2.2, and for $s=3$, we have $\bar{w}_{16}(G_{9,4}) \neq 0$, as more generally $\bar{w}_{k^2}(G_{2k+1,k}) \neq 0$ by [4]. Part (iv) is a special case of (i) (take $s+1$ instead of s).

Results known up to now do not cover the Grassmann manifolds $G_{n,4}$ with $n \in \{2^s+2, 2^s+3\}$. For these, we shall prove the following theorem, anticipated by Professor Koichi Iwata in a conjecture, which he communicated to the first named author in 1996.

THEOREM 2.4. *Let* $s \geqq 3$.
(i) *If* $n = 2^s + 2$, *then* $\bar{w}_q(G_{n,4}) \neq 0$ *for* $q = 2^{s+1} + 4$;
(ii) *if* $n = 2^s + 3$, *then* $\bar{w}_q(G_{n,4}) \neq 0$ *for* $q = 3 \cdot 2^s$.

REMARKS 2.5. (a) Taking the 2nd exterior power of $\gamma \oplus \gamma^\perp \cong \varepsilon^{10}$ over $G_{10,4}$, we readily see that $TG_{10,4} \oplus \lambda^2(\gamma) \oplus \lambda^2(\gamma^\perp) \cong \varepsilon^{45}$, thus $G_{10,4}$ immerses in $\mathbb{R}^{45}$. At the same time, Theorem 2.4(i) implies that $G_{10,4}$ does not immerse in $\mathbb{R}^{43}$ and, more generally, $G_{2^s+2,4}$ does not immerse in $\mathbb{R}^{6 \cdot 2^s - 5}$ for $s \geqq 3$. For $s = 3$, as we have seen, this non-immersion result can be improved by no more than one, and is better than the non-immersion result implied by [7, Proposition 5.1], attained by different methods. For $s > 3$, the non-immersion dimension of [7] is better, but our $\bar{w}_q(G_{n,4}) \neq 0$ gives interesting geometric information (not provided by [7]) about the normal bundle of any immersion of $G_{n,4}$ in Euclidean space. We also note that Theorem 2.4(ii) implies that $G_{2^s+3,4}$ does not immerse in $\mathbb{R}^{7 \cdot 2^s - 5}$.

(b) Another application of our result is the nonexistence of $2t$-regular maps from $G_{2^s+2,4}$ to $\mathbb{R}^{3t(2^{s+1}-1)}$ and from $G_{2^s+3,4}$ to $\mathbb{R}^{t(7 \cdot 2^s - 3)}$. Recall that (see [3]) there exists a $2t$-regular map from $G_{2^s+2,4}$ to $\mathbb{R}^{2t(2^{s+2}-7)+1}$ and from $G_{2^s+3,4}$ to $\mathbb{R}^{2t(2^{s+2}-3)+1}$.

3. Proof of Theorem 2.4

PROOF OF 2.4(i). To prove that $\bar{w}_q(G_{2^s+2,4}) \neq 0$ for $q = 2^{s+1} + 4$, it suffices to show that the cohomology class

$$\bar{w}_q(G_{2^s+2,4}) \cdot w_2^{2^s-8} w_4 \tag{3.1}$$

in the cohomology group $H^{\text{top}}(G_{2^s+2,4}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ does not vanish. By [11, Lemma 1, p. 107], the value of (3.1) is the same (zero or nonzero) as the value of

$$i^*\big(\bar{w}_q(G_{2^s+2,4}) \cdot w_2^{2^s-8}\big) \tag{3.2}$$

in $H^{\text{top}}(G_{2^s+1,4}; \mathbb{Z}_2)$, where $i : G_{2^s+1,4} \to G_{2^s+2,4}$ is induced by the inclusion $\mathbb{R}^{2^s+1} \hookrightarrow \mathbb{R}^{2^s+2}$. Since $i^*(\gamma) = \gamma$ and $i^*(\gamma^\perp) = \gamma^\perp \oplus \varepsilon^1$, using (1.1), we transform (3.2) to

$$\begin{aligned} i^*\big(w_q\big(\gamma \otimes \gamma \oplus (2^s+2)\gamma^\perp\big) \cdot w_2^{2^s-8}\big) &= \big[w(\gamma \otimes \gamma) w(\gamma^\perp)^{2^s+2}\big]_q \cdot w_2^{2^s-8} \\ &= \big[w(\gamma \otimes \gamma)(1 + \bar{w}_1^{2^s} + \cdots + \bar{w}_{2^s-3}^{2^s}) w(\gamma^\perp)^2\big]_q \cdot w_2^{2^s-8}. \end{aligned}$$

We shall need the following auxiliary result.

LEMMA 3.3. *For $k \geqq 2$, $\bar{w}_k^{2^s} = 0$ in $H^*(G_{2^s+1,4};\mathbb{Z}_2)$.*

PROOF. It suffices to show that $\bar{w}_k^{2^s} \cdot x = 0$ for every x such that $\bar{w}_k^{2^s} \cdot x \in H^{\mathrm{top}}(G_{2^s+1,4};\mathbb{Z}_2)$. To show this, it suffices (see Fact (c)) to prove that

$$\pi^*(\bar{w}_k^{2^s} \cdot x) e_1^3 e_2^2 e_3 e_5^{2^s-4} e_6^{2^s-5} \dots e_{2^s-1}^2 e_{2^s} \tag{3.4}$$

in $H^{\mathrm{top}}\big(\mathrm{Flag}\,(\mathbb{R}^{2^s+1});\mathbb{Z}_2\big)$ vanishes. The factor $\pi^*(\bar{w}_k^{2^s})$ is nothing but the 2^sth power of the kth elementary symmetric function in the variables $e_5, e_6, \dots, e_{2^s+1}$. Since $k \geqq 2$, in every monomial $e_1^{i_1} e_2^{i_2} \dots e_{2^s+1}^{i_{2^s+1}}$ of (3.4), there is at least one e_j with the exponent $i_j \geqq 2^s+1$. Hence the class (3.4) vanishes (see Fact (d) or [11, Corollary, p. 106]). □

Using Lemma 3.3 and the identity $\bar{w}_1 = w_1$, one can transform (3.2) to

$$\begin{aligned} i^*\big(\bar{w}_q(G_{2^s+2,4}) \cdot w_2^{2^s-8}\big) &= \big[w(\gamma\otimes\gamma)(1+w_1^{2^s})w(\gamma^\perp)^2\big]_q \cdot w_2^{2^s-8} \\ &= \big[w(\gamma\otimes\gamma)w(\gamma^\perp)^2\big]_q \cdot w_2^{2^s-8} + \big[w(\gamma\otimes\gamma)w(\gamma^\perp)^2 w_1^{2^s}\big]_q \cdot w_2^{2^s-8}. \end{aligned} \tag{3.5}$$

We now prove another auxiliary result.

LEMMA 3.6. *In $H^*(G_{2^s+1,4};\mathbb{Z}_2)$, we have*

$$\big[w(\gamma\otimes\gamma)w(\gamma^\perp)^2 w_1^{2^s}\big]_{2^{s+1}+4} \cdot w_2^{2^s-8} = 0.$$

PROOF. In fact, we want to prove that

$$\big[w(\gamma\otimes\gamma)w(\gamma^\perp)^2\big]_{2^s+4} \cdot w_1^{2^s} \cdot w_2^{2^s-8} \tag{3.7}$$

vanishes. Let $\tilde{i} : G_{2^s-1,3} \to G_{2^s+1,4}$ be the inclusion obtained as the composition $\tilde{i} = l \circ j$, where $j : G_{2^s-1,3} \to G_{2^s,3}$ and $l : G_{2^s,3} \to G_{2^s+1,4}$ are the "standard" inclusions, i.e., we have $j^*(\gamma) = \gamma$, $j^*(\gamma^\perp) = \gamma^\perp \oplus \varepsilon^1$, $l^*(\gamma) = \gamma \oplus \varepsilon^1$, $l^*(\gamma^\perp) = \gamma^\perp$. Hence $\tilde{i}^*(\gamma) = \gamma \oplus \varepsilon$, $\tilde{i}^*(\gamma^\perp) = \gamma^\perp \oplus \varepsilon$, and also $\tilde{i}^*(w_k) = w_k$, $\tilde{i}^*(\bar{w}_k) = \bar{w}_k$ for every k. By [11, Lemma 3, p. 108], the value of (3.7) is the same as the value of

$$\begin{aligned} &\tilde{i}^*\Big(\big[w(\gamma\otimes\gamma)w(\gamma^\perp)^2\big]_{2^s+4} \cdot w_2^{2^s-8}\Big) \\ &= \big[w\big((\gamma\oplus\varepsilon^1)\otimes(\gamma\oplus\varepsilon^1)\big)\,w(\gamma^\perp)^2\big]_{2^s+4} \cdot w_2^{2^s-8} \\ &= \big[w(\gamma\otimes\gamma)w(\gamma)^2 w(\gamma^\perp)^2\big]_{2^s+4} \cdot w_2^{2^s-8} = \big[w(\gamma\otimes\gamma)\big]_{2^s+4} \cdot w_2^{2^s-8}. \end{aligned}$$

But the dimension of the bundle $\gamma\otimes\gamma$ in the latter is 9, hence $w_{2^s+4}(\gamma\otimes\gamma) = 0$ (we have $s \geqq 3$), which implies the lemma. □

By Lemma 3.6, the second part of (3.5) vanishes. Hence, to show that (3.2) does not vanish, we need to show that

$$(3.8)\quad \left[w(\gamma\otimes\gamma)w(\gamma^{\perp})^2\right]_q\cdot w_2^{2^s-8} = \left[w(\gamma\otimes\gamma)\left(1+\bar w_1^2+\dots+\bar w_{2^s-3}^2\right)\right]_q\cdot w_2^{2^s-8}$$

in $H^*(G_{2^s+1,4};\mathbb{Z}_2)$ does not vanish.

By [1], [6], we have

$$w(\gamma\otimes\gamma) = 1+w_1^2+w_1^4+w_1^6+\left(w_1^2w_3^2+w_2^4\right) + \left(w_1^2w_2^4+w_1^4w_3^2\right)+\left(w_1^4w_4^2+w_1^2w_2^2w_3^2+w_3^4\right).$$

Obviously, $w_j(\gamma\otimes\gamma)\left(1+\bar w_1^2+\ldots+\bar w_{2^s-3}^2\right)$ vanishes for $j\leqq 8$ in dimension $2^{s+1}+4$, hence (3.8) can be written as

$$\begin{aligned} w_2^{2^s-8}&\left(w_{10}(\gamma\otimes\gamma)\bar w_{2^s-3}^2+w_{12}(\gamma\otimes\gamma)\bar w_{2^s-4}^2\right)\\ &= \left(w_1^2w_2^{2^s-4}+w_1^4w_2^{2^s-8}w_3^2\right)\bar w_{2^s-3}^2\\ &+\left(w_1^4w_2^{2^s-8}w_4^2+w_1^2w_2^{2^s-6}w_3^2+w_2^{2^s-8}w_3^4\right)\bar w_{2^s-4}^2. \end{aligned}$$

To show that the latter is nonzero, we proceed in two steps (using Fact (c) or [11, Observation, p. 106]).

Step (a). We verify that

$$A := \pi^*\left(\left(w_1^2w_2^{2^s-4}+w_1^4w_2^{2^s-8}w_3^2\right)\bar w_{2^s-3}^2\right)e_1^3e_2^2e_3e_5^{2^s-4}e_6^{2^s-5}\ldots e_{2^s}^1\neq 0.$$

Since $\pi^*(\bar w_{2^s-3}^2)=e_5^2e_6^2\ldots e_{2^s+1}^2$, we have

$$A=\pi^*\left(w_1^2w_2^{2^s-4}+w_1^4w_2^{2^s-8}w_3^2\right)e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\ldots e_{2^s}^3e_{2^s+1}^2.$$

Recall (Fact (b)) that $\pi^*(w_i)$ is the ith elementary symmetric function in e_1, e_2, e_3, e_4. In

$$A_1 := \pi^*\left(w_1^2w_2^{2^s-4}\right)e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\ldots e_{2^s}^3e_{2^s+1}^2$$

expressed in terms of $e_1,e_2,\ldots,e_{2^s+1}$ (note that $e_5^{2^s-2}e_6^{2^s-3}\ldots e_{2^s}^3e_{2^s+1}^2$ is precisely in this form in every monomial in A_1), all the nonzero monomials must not contain the factor e_4 and must have e_3 in the first power. Hence the contribution of $\pi^*(w_1^2w_2^{2^s-4})$ to the nonzero monomials must not contain e_3 nor e_4, and we have

$$A_1=(e_1+e_2)^2(e_1e_2)^{2^s-4}e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\ldots e_{2^s}^3e_{2^s+1}^2$$

$$= \underbrace{e_1^{2^s+1}e_2^{2^s-2}e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3e_{2^s+1}^2}_{=0}$$

$$+ \underbrace{e_1^{2^s-1}e_2^{2^s}e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3e_{2^s+1}^2}_{\neq 0} \neq 0.$$

The second part of A is given by

$$A_2 := \pi^*\big(w_1^4w_2^{2^s-8}w_3^2\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3e_{2^s+1}^2$$
$$= \pi^*\big(w_1^4w_2^{2^s-8}\big)\,\big(e_1^2e_2^2e_3^2 + e_1^2e_2^2e_4^2 + e_1^2e_3^2e_4^2$$
$$+ e_2^2e_3^2e_4^2\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3e_{2^s+1}^2.$$

Here every monomial either contains e_4 (thus does not contain any e_j with the exponent zero), or contains e_3 with the exponent greater than 1 (thus does not contain any e_j in the first power). Hence $A_2 = 0$ and $A = A_1 + A_2 \neq 0$. □

Step (b). We verify that

$$B := \pi^*\Big(\big(w_1^4w_2^{2^s-8}w_4^2 + w_1^2w_2^{2^s-6}w_3^2$$
$$+ w_2^{2^s-8}w_3^4\big)\,\bar{w}_{2^s-4}^2\Big)e_1^3e_2^2e_3e_5^{2^s-4}e_6^{2^s-5}\dots e_{2^s}^1 = 0.$$

Since

$$\pi^*(\bar{w}_{2^s-4}^2) = e_5^2e_6^2\dots e_{2^s}^2 + e_5^2e_6^2\dots e_{2^s-1}^2e_{2^s+1}^2 + e_5^2e_6^2\dots e_{2^s-2}^2e_{2^s}^2e_{2^s+1}^2$$
$$+ e_5^2e_6^2\dots e_{2^s-3}^2e_{2^s-1}^2e_{2^s}^2e_{2^s+1}^2 + \dots + e_5^2e_7^2\dots e_{2^s}^2e_{2^s+1}^2 + e_6^2e_7^2\dots e_{2^s}^2e_{2^s+1}^2,$$

we have

$$B = \pi^*\big(w_1^4w_2^{2^s-8}w_4^2 + w_1^2w_2^{2^s-6}w_3^2 + w_2^{2^s-8}w_3^4\big)\,e_1^3e_2^2e_3$$
$$\cdot\big(e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3 + e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s-1}^4e_{2^s}^1e_{2^s+1}^2$$
$$+ e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s-2}^5e_{2^s-1}^{\boxed{2}}e_{2^s}^3e_{2^s+1}^{\boxed{2}}$$
$$+ e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s-3}^6e_{2^s-2}^{\boxed{3}}e_{2^s-1}^4e_{2^s}^{\boxed{3}}e_{2^s+1}^2 + \cdots$$
$$\cdots + e_5^{2^s-2}e_6^{\boxed{2^s-5}}e_7^{2^s-4}e_8^{\boxed{2^s-5}}e_9^{2^s-6}\dots e_{2^s}^3e_{2^s+1}^2$$
$$+ e_5^{\boxed{2^s-4}}e_6^{2^s-3}e_7^{\boxed{2^s-4}}e_8^{2^s-5}\dots e_{2^s}^3e_{2^s+1}^2\big).$$

After expressing $\pi^*(w_1^4w_2^{2^s-8}w_4^2+w_1^2w_2^{2^s-6}w_3^2+w_2^{2^s-8}w_3^4)$ in terms of e_1, e_2, e_3, e_4, those parts of monomials containing powers of $e_5, e_6, \dots, e_{2^s+1}$ do not change, hence only the first two summands in the large parentheses may contribute to the nonzero monomials, while in the others, some of the exponents are repeated. Thus we have

$$B=\underbrace{\pi^*\big(w_1^4w_2^{2^s-8}w_4^2+w_1^2w_2^{2^s-6}w_3^2+w_2^{2^s-8}w_3^4\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3}_{:=B_1}$$

$$+\underbrace{\pi^*\big(w_1^4w_2^{2^s-8}w_4^2+w_1^2w_2^{2^s-6}w_3^2+w_2^{2^s-8}w_3^4\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s-1}^4e_{2^s}^1e_{2^s+1}^2}_{:=B_2}.$$

Any nonzero monomial in B_1 must have e_3 in the first power. Hence all the monomials coming from

$$\pi^*(w_1^4w_2^{2^s-8}w_4^2)e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$$
$$=\pi^*(w_1^4w_2^{2^s-8})e_1^2e_2^2e_3^2e_4^2e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$$

vanish. For the same reason, the only candidates for nonzero monomials coming from $\pi^*(w_2^{2^s-8}w_3^4)e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$ are those coming from

$$(e_1e_2+e_1e_4+e_2e_4)^{2^s-8}e_1^4e_2^4e_4^4\cdot e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3.$$

In every such monomial, the exponent of e_4 is the same as one of the exponents in $e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$ (the exponent j of e_4 satisfies $4\leqq j\leqq 2^s-4$). Hence

$$\pi^*\big(w_2^{2^s-8}w_3^4\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3=0.$$

Also the third part of B_1, having the form $\pi^*\big(w_1^2w_2^{2^s-6}w_3^2\big)\,e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$, may have nonzero monomials only with e_3 in the first power. Such monomials come only from

$$\pi^*\big(w_1^2w_2^{2^s-6}\big)\,e_1^2e_2^2e_4^2\cdot e_1^3e_2^2e_3e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$$
$$=\pi^*\big(w_1^2w_2^{2^s-6}\big)\,e_1^5e_2^4e_3e_4^2e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3.$$

Here the nonzero monomials must contain $e_3e_4^2$ (otherwise one cannot "produce" the exponents 1 and 2), so the only candidates for nonzero monomials are those two obtained from

$$(e_1+e_2)^2(e_1e_2)^{2^s-6}e_1^5e_2^4e_3e_4^2e_5^{2^s-2}e_6^{2^s-3}\dots e_{2^s}^3$$

$$= \underbrace{e_1^{2^s+1} e_2^{2^s-2} e_3 e_4^2 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s}^3}_{=0} + \underbrace{e_1^{2^s-1} e_2^{2^s} e_3 e_4^2 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s}^3}_{\neq 0} \neq 0.$$

Hence we have $B_1 \neq 0$.

Now we determine the value of B_2. Obviously, any nonzero monomial in it must not contain e_4 (it contains all the other classes e_j). Hence the contribution of

$$\pi^*\big(w_1^4 w_2^{2^s-8} w_4^2\big)\, e_1^3 e_2^2 e_3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$$
$$= \pi^*\big(w_1^4 w_2^{2^s-8}\big)\, e_1^2 e_2^2 e_3^2 e_4^2 \cdot e_1^3 e_2^2 e_3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$$

is zero. The contribution of $\pi^*\big(w_2^{2^s-8} w_3^4\big)\, e_1^3 e_2^2 e_3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$ is also zero. Indeed, its nonzero monomials might only come from

$$\pi^*\big(w_2^{2^s-8}\big)\, e_1^4 e_2^4 e_3^4 \cdot e_1^3 e_2^2 e_3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$$
$$= \pi^*\big(w_2^{2^s-8}\big)\, e_1^7 e_2^6 e_3^5 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2,$$

but here, in every monomial, there is no e_j with exponent 3, hence every monomial vanishes. We are left with the class $\pi^*\big(w_1^2 w_2^{2^s-6} w_3^2\big)\, e_1^3 e_2^2 e_3 e_5^{2^s-2}$ $e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$. Since any nonzero monomial must not contain e_4 in this, it may only come from

$$\pi^*\big(w_1^2 w_2^{2^s-6}\big)\, e_1^2 e_2^2 e_3^2 \cdot e_1^3 e_2^2 e_3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$$
$$= \pi^*\big(w_1^2 w_2^{2^s-6}\big)\, e_1^5 e_2^4 e_3^3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2.$$

Here, in nonzero monomials, we must have e_3^3, so the only candidates for nonzero monomials are those two obtained from

$$(e_1+e_2)^2 (e_1 e_2)^{2^s-6} e_1^5 e_2^4 e_3^3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2$$
$$= \underbrace{e_1^{2^s+1} e_2^{2^s-2} e_3^3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2}_{=0}$$
$$+ \underbrace{e_1^{2^s-1} e_2^{2^s} e_3^3 e_5^{2^s-2} e_6^{2^s-3} \dots e_{2^s-1}^4 e_{2^s}^1 e_{2^s+1}^2}_{\neq 0} \neq 0.$$

Hence $B_2 \neq 0$ and we obtain $B = B_1 + B_2 = 0$. □

We have shown that the value of (3.2) is $A + B \neq 0$, hence (3.1) does not vanish, and Theorem 2.4(i) is proven. □

An Outline of the Proof of 2.4(ii). To prove that $\bar{w}_q(G_{2^s+3,4}) \neq 0$ for $q = 3 \cdot 2^s$, it suffices to show that the cohomology class

$$\bar{w}_q(G_{2^s+3,4}) \cdot w_1^{2^s-8} w_4 \tag{3.9}$$

in the cohomology group $H^{\mathrm{top}}(G_{2^s+3,4};\mathbb{Z}_2) \cong \mathbb{Z}_2$ does not vanish. In the same way as in part (i), we have that the value of (3.9) is the same (zero or nonzero) as the value of

$$V := i^*\big(\bar{w}_q(G_{2^s+3,4}) \cdot w_1^{2^s-8}\big) = \big[w(\gamma\otimes\gamma)w(\gamma^\perp)^{2^s+3}\big]_q \cdot w_1^{2^s-8} \tag{3.10}$$

in $H^{\mathrm{top}}(G_{2^s+2,4};\mathbb{Z}_2)$. Since $\dim(G_{2^s+2,4}) = 4(2^s-2) < 2\cdot 2^{s+1}$, we have $w_i^{2^{s+1}} = 0$ for $i = 2,3,4$. By [11], in $G_{2^s+2,4}$, we have height $(w_1) = 2^{s+1}-1$, hence $w_1^{2^{s+1}} = 0$ and $w(\gamma)^{2^{s+1}} = (1+w_1+w_2+w_3+w_4)^{2^{s+1}} = 1$. We can transform (3.10) to

$$\begin{aligned} V &= \big[w(\gamma\otimes\gamma)w(\gamma^\perp)^{2^s+3}\big]_q \cdot w_1^{2^s-8} \\ &= \big[w(\gamma\otimes\gamma)w(\gamma^\perp)^{2^s+3}w(\gamma)^{2^{s+1}}\big]_q \cdot w_1^{2^s-8} \\ &= \big[w(\gamma\otimes\gamma)w(\gamma)^{2^s-3}\big]_q \cdot w_1^{2^s-8} = \Big[w(\gamma\otimes\gamma)w(\gamma)\cdot\prod_{i=2}^{s-1} w(\gamma)^{2^i}\Big]_q \cdot w_1^{2^s-8}. \end{aligned}$$

Let us denote $P := w(\gamma\otimes\gamma)w(\gamma)\cdot\prod_{i=2}^{s-2} w(\gamma)^{2^i}$ (for $s=3$, we put $\prod_{i=2}^{1} w(\gamma)^{2^i} = 1$). Then we can write

$$\begin{aligned} V = {} & \underbrace{[P]_{3\cdot 2^s}\cdot w_1^{2^s-8}}_{:=A} + \underbrace{[P]_{5\cdot 2^{s-1}}\cdot w_1^{2^{s-1}}\cdot w_1^{2^s-8}}_{:=B} + \underbrace{[P]_{2^{s+1}}\cdot w_2^{2^{s-1}}\cdot w_1^{2^s-8}}_{:=C} \\ & + \underbrace{[P]_{3\cdot 2^{s-1}}\cdot w_3^{2^{s-1}}\cdot w_1^{2^s-8}}_{:=D} + \underbrace{[P]_{2^s}\cdot w_4^{2^{s-1}}\cdot w_1^{2^s-8}}_{:=E}. \end{aligned}$$

Since $w_i(\gamma\otimes\gamma) = 0$ for $i \geqq 13$, we know that $[P]_k = 0$ for all k exceeding $12+4+4(4+8+\cdots+2^{s-2}) = 2^{s+1}$. So $[P]_{3\cdot 2^s} = [P]_{5\cdot 2^{s-1}} = 0$, and $A = B = 0$. It remains to find the values of C, D, E. Again, one uses Fact (c). We take V as a polynomial in w_1, w_2, w_3, and w_4; therefore any nonzero monomial in $\pi^*(u)\cdot e_1^3e_2^2e_3e_5^{2^s-3}e_6^{2^s-4}\dots e_{2^s+1}$ is of the form $e_1^{i_1}e_2^{i_2}e_3^{i_3}e_4^{i_4}e_5^{2^s-3}e_6^{2^s-4}\dots e_{2^s+1}$, where $\{i_1,i_2,i_3,i_4\} = \{2^s-2, 2^s-1, 2^s, 2^s+1\}$. Of course, the top-dimension component of P is $[P]_{2^{s+1}} = w_{12}(\gamma\otimes\gamma)\cdot w_4^{1+4+8+\cdots+2^{s-2}}$. Since $w_{12}(\gamma\otimes\gamma) = w_1^4w_4^2 + w_1^2w_2^2w_3^2 + w_3^4$, we obtain

$$C = \big(w_1^4w_4^2 + w_1^2w_2^2w_3^2 + w_3^4\big)\, w_4^{2^{s-1}-3} w_2^{2^{s-1}} w_1^{2^s-8}$$

$$= \underbrace{w_1^{2^s-4}w_2^{2^{s-1}}w_4^{2^{s-1}-1}}_{:=C_1} + \underbrace{w_1^{2^s-6}w_2^{2^{s-1}+2}w_3^2w_4^{2^{s-1}-3}}_{:=C_2} + \underbrace{w_1^{2^s-8}w_2^{2^{s-1}}w_3^4w_4^{2^{s-1}-3}}_{:=C_3}.$$

We have

$$\pi^*(C_1)e_1^3e_2^2e_3 = (e_1+e_2+e_3+e_4)^{2^s-4}$$
$$\cdot(e_1e_2+e_1e_3+e_1e_4+e_2e_3+e_2e_4+e_3e_4)^{2^{s-1}}(e_1e_2e_3e_4)^{2^{s-1}-1}e_1^3e_2^2e_3$$
$$= e_1^{2^{s-1}+2}e_2^{2^{s-1}+1}e_3^{2^{s-1}}e_4^{2^{s-1}-1}(e_1+e_2+e_3+e_4)^{2^{s-1}-4}$$
$$\cdot\big((e_1+e_2+e_3+e_4)(e_1e_2+e_1e_3+e_1e_4+e_2e_3+e_2e_4+e_3e_4)\big)^{2^{s-1}}$$
$$= e_1^{2^{s-1}+2}e_2^{2^{s-1}+1}e_3^{2^{s-1}}e_4^{2^{s-1}-1}(e_1+e_2+e_3+e_4)^{2^{s-1}-4}$$
$$\cdot\Big(\sum e_i^{2^s}e_j^{2^{s-1}} + \big(e_1^{2^{s-1}}e_2^{2^{s-1}}e_3^{2^{s-1}} + e_1^{2^{s-1}}e_2^{2^{s-1}}e_4^{2^{s-1}}$$
$$+ e_1^{2^{s-1}}e_3^{2^{s-1}}e_4^{2^{s-1}} + e_2^{2^{s-1}}e_3^{2^{s-1}}e_4^{2^{s-1}}\big)\Big).$$

From the last parentheses, neither $e_i^{2^s}e_j^{2^{s-1}}$ (the exponent of e_i would grow to at least $2^s+2^{s-1}-1>2^s+1$), nor $e_1^{2^{s-1}}e_j^{2^{s-1}}e_k^{2^{s-1}}$ (the exponent of e_1 would grow to at least $2^{s-1}+2^{s-1}+2>2^s+1$) can contribute to nonzero monomials in the top dimension. We are left with

$$e_1^{2^{s-1}+2}e_2^{2^{s-1}+1}e_3^{2^{s-1}}e_4^{2^{s-1}-1}(e_1+e_2+e_3+e_4)^{2^{s-1}-4}e_2^{2^{s-1}}e_3^{2^{s-1}}e_4^{2^{s-1}}$$
$$= e_1^{2^{s-1}+2}e_2^{2^s+1}e_3^{2^s}e_4^{2^s-1}(e_1+e_2+e_3+e_4)^{2^{s-1}-4}.$$

To obtain a nonzero monomial in the top dimension, we must not increase the exponents of e_2, e_3, e_4 after multiplying $e_1^{2^{s-1}+2}e_2^{2^s+1}e_3^{2^s}e_4^{2^s-1}$ by the last parentheses. So we conclude that

$$\pi^*(C_1)e_1^3e_2^2e_3e_5^{2^s-3}\dots e_{2^s+1} = e_1^{2^{s-1}+2}e_2^{2^s+1}e_3^{2^s}e_4^{2^s-1}e_1^{2^{s-1}-4}e_5^{2^s-3}\dots e_{2^s+1}$$
$$= e_1^{2^s-2}e_2^{2^s+1}e_3^{2^s}e_4^{2^s-1}e_5^{2^s-3}\dots e_{2^s+1},$$

which is nonzero, and therefore also $C_1 \neq 0$.

In an analogous way, one proves that

$$C_2 = C_3 = 0 \quad \text{for } s>3, \qquad C_2, C_3 \neq 0 \quad \text{for } s=3,$$

and so one obtains that (for any $s \geqq 3$) $C = C_1 + (C_2+C_3) = C_1 + 0 \neq 0$.

Similarly to the above, one can show that

$$D = E = 0 \quad \text{for } s>3, \qquad D, E \neq 0 \quad \text{for } s=3;$$

details are omitted.

Finally, we have $V = A+B+C+(D+E) = 0+0+C+0 \neq 0$, which proves Theorem 2.4(ii). □

References

[1] V. Bartík and J. Korbaš, Stiefel–Whitney characteristic classes and parallelizability of Grassmann manifolds, *Rend. Circ. Mat. Palermo (2)*, **33** (1984), 19–29.

[2] A. Borel, La cohomologie mod 2 de certains espaces homogènes, *Comment. Math. Helv.*, **27** (1953), 165–197.

[3] D. Handel, $2k$-regular maps on smooth manifolds, *Proc. Amer. Math. Soc.*, **124** (1996), 1609–1613.

[4] H. Hiller and R. E. Stong, Immersion dimension for real Grassmannians, *Math. Ann.*, **255** (1981), 361–367.

[5] W. C. Hsiang and R. H. Szczarba, On the tangent bundle of a Grassman manifold, *Amer. J. Math.*, **86** (1964), 698–704.

[6] J. Korbaš, On the Stiefel–Whitney classes and the span of real Grassmannians, *Czechoslovak Math. J.*, **36(111)** (1986), 541–552.

[7] K. H. Mayer, Nonimmersion theorems for Grassmann manifolds, *Topology Appl.*, **96** (1999), 171–184.

[8] K. G. Monks, Groebner bases on the cohomology of Grassmann manifolds with application to immersion, *Bol. Soc. Mat. Mexicana (3)*, **7** (2001), 123–136.

[9] V. Oproiu, Some non-embedding theorems for the Grassmann manifolds $G_{2,n}$ and $G_{3,n}$, *Proc. Edinburgh Math. Soc.*, **20** (1976-77), 177–185.

[10] V. Oproiu, Some results concerning the non-embedding codimension of Grassmann manifolds in Euclidean spaces, *Rev. Roumaine Math. Pures Appl.*, **26** (1981), 275–286.

[11] R. E. Stong, Cup products in Grassmannians, *Topology Appl.*, **13** (1982), 103–113.

[12] R. M. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer-Verlag (Berlin, 1975).